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THE HEAT EQUATION WITH A SINGULAR POTENTIAL

BY

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ABSTRACT. Of concern is the singular problem $\partial u/\partial t = \Delta u + (c/|x|^2)u + f(t, x)$, $u(x, 0) = u_0(x)$, and its generalizations. Here $c \geqslant 0$, $x \in \mathbf{R}^N$, $t > 0$, and f and u_0 are nonnegative and not both identically zero. There is a dimension dependent constant $C_*(N)$ such that the problem has no solution for $c > C_*(N)$. For $c \leqslant C_*(N)$ necessary and sufficient conditions are found for f and u_0 so that a nonnegative solution exists.

1. Introduction. Of concern is the heat equation with a potential

$$\partial u/\partial t - \Delta u = V(x)u + f(x, t)$$

for $t > 0$ and $x \in \Omega \subset \mathbf{R}^N$. Take either $\Omega = \mathbf{R}^N$ or else Ω to be a bounded domain containing $B_1 = \{x \in \mathbf{R}^N: |x| < 1\}$, in which case we impose the Dirichlet boundary condition

$$u(x, t) = 0 \quad \text{for } x \in \partial\Omega.$$

The initial condition is

$$u(x, 0) = u_0(x), \qquad x \in \Omega.$$

We take $u_0, f \geqslant 0$ and $0 \leqslant V \in L^\infty(\Omega \setminus B_\varepsilon)$ (where $B_\varepsilon = \{x: |x| < \varepsilon\}$) for each $\varepsilon > 0$, but V is singular at the origin. The question is: How singular must V be to prevent a solution u from existing? The answer, informally stated, is that V is too singular if $V(x) > C_*(N)/|x|^2$ near $x = 0$, while V is not too singular if $V(x) \leqslant C_*(N)/|x|^2$ near $x = 0$, where $C_*(N) = ((N-2)/2)^2$, for $N = 1, 2, 3, \ldots$.

Let us be more precise. Let Ω be a domain in $\mathbf{R}^N$ with $B_1 \subset \Omega \subset \mathbf{R}^N$. If $\Omega \neq \mathbf{R}^N$ let the boundary of Ω be nice enough. Let

$$V_*(x) = c/|x|^2, \qquad x \in \Omega,$$

where $c > 0$. Consider the problem

$$\text{(Q)} \qquad \begin{aligned} &\partial u/\partial t - \Delta u = V_* u \quad \text{in } \Omega \times]0, T[, \\ &u = 0 \quad \text{on } \partial\Omega \times]0, T[, \\ &u(0, x) = u_0(x) \quad \text{in } \Omega \end{aligned}$$

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with $u_0 \geqslant 0$ and $u_0 \not\equiv 0$ (i.e. u_0 not equal to zero a.e.). Here $0 < T \leqslant \infty$. Because V is so singular at the origin it is not clear if a solution u of (Q) exists. So we approximate V_* by

$$V_n(x) = \min\{V_*(x), n\},$$

and we let u_n be the unique nonnegative solution of

$$(\mathrm{Q}_n) \qquad \begin{aligned} &\partial u/\partial t - \Delta u = V_n u_n \quad \text{in } \Omega \times]0, T[, \\ &u_n = 0 \quad \text{on } \partial\Omega \times]0, T[, \\ &u_n(0, x) = u_0(x) \quad \text{in } \Omega. \end{aligned}$$

Note that u_n exists if u_0 satisfies some innocuous conditions, which we assume. Let

$$C_*(N) = ((N-2)/2)^2.$$

It will follow from the results of this paper that, if u_n is the solution of (Q_n), then:

(I) If $0 \leqslant c \leqslant C_*(N)$, then $\lim_{n\to\infty} u_n(x, t) = u(x, t)$ exists and u is a solution of (Q).

(II) If $c > C_*(N)$, then $\lim_{n\to\infty} u_n(x, t) = \infty$ for all $(x, t) \in \Omega \times]0, T[$.

In the existence result (I), by an argument using the maximum principle we can replace V_n, V_* by $\tilde{V}_n$, $\tilde{V}_*$, respectively, where $\tilde{V}_n(x) \leqslant V_n(x)$ a.e. for each n. Similarly, in the nonexistence result (II) we can replace V_n, V_* by $\tilde{V}_n$, $\tilde{V}_*$ where $\tilde{V}_n \geqslant V_n$ a.e. for each n (or at least $\tilde{V}_n \geqslant V_n$ a.e. in a fixed neighborhood of the origin).

Two proofs will be given of the nonexistence result. One, based on the techniques of the theory of partial differential equations, is closely related to the existence results such as (I). The other, for the case of $\Omega = \mathbf{R}^N$, is probabilistic and is based on the Feynman-Kac integral formula. This represents, to our knowledge, the first such application of the Feynman-Kac formula to a nonexistence question in partial differential equations.

In the next section the main results are stated. §§3–5 contain the proofs based on the techniques of partial differential equations. The probabilistic proof is given in §6. §7 contains complements and remarks.

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2. Statements of the main results. Let Ω be a domain satisfying $B_1 \subset \Omega \subset \mathbf{R}^N$. If $N = 1$ we delete the origin so that $0 \notin \Omega$; for simplicity we may take $\Omega =]0, R[$ where $1 \leqslant R \leqslant \infty$. Let $0 \leqslant V \in L^1_{\mathrm{loc}}(\Omega)$, let $0 \leqslant f \in L^1(\Omega \times]0, T[)$, and let u_0 be a nonnegative function in $L^1(\Omega)$, or, more generally, let u_0 be a finite (positive) Radon measure on Ω. Consider the problem of finding a function u such that

$$(\mathrm{P}) \qquad \begin{aligned} &u \geqslant 0 \text{ on } \Omega \times (0, T), \qquad Vu \in L^1_{\mathrm{loc}}(\Omega \times]0, T[), \\ &\partial u/\partial t - \Delta u = Vu + f \quad \text{in } \mathcal{D}'(\Omega \times]0, T[), \\ &\operatorname*{ess\,lim}_{t\to 0^+} \int_\Omega u(t)\phi = \int_\Omega \phi u_0 \quad \text{for all } \phi \in \mathcal{D}(\Omega). \end{aligned}$$

Here $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ with the usual topology; $\mathcal{D}'$, the dual space of $\mathcal{D}$, is a space of distributions; and for typographical convenience, arguments of functions and differentials in integrals will be omitted when no confusion can arise.

We shall attack (P) by studying the approximate problem

$$(\mathrm{P}_n) \qquad \begin{aligned} &\partial u_n/\partial t - \Delta u_n = V_n u_n + f_n \quad \text{in } \mathcal{D}'(\Omega \times]0, T[), \\ &u_n = 0 \quad \text{on } \partial\Omega \text{ for all } t \in]0, T[, \\ &\lim_{t\to 0} \int_\Omega u_n(t)\phi = \int \phi u_0 \quad \text{for all } \phi \in \mathcal{D}(\Omega), \end{aligned}$$

where $f_n = \min\{f, n\}$ and where $V_n \in L^\infty(\Omega)$, $0 \leqslant V_n \leqslant V$, and $V_n \uparrow V$ a.e. in Ω. Of course, the Dirichlet boundary condition will be absent if $\Omega = \mathbf{R}^N$.

The problem (P_n) has a unique bounded nonnegative solution which satisfies the integral equation

$$u_n(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}V_n u_n(s)\,ds + \int_0^t e^{(t-s)\Delta}f_n(s)\,ds, \tag{2.1}$$

where $\{e^{t\Delta}\colon t \geqslant 0\}$ denotes the semigroup generated by Δ with Dirichlet boundary conditions; note that the perturbation V_n defines a bounded multiplication operator on $L^p(\Omega)$ for all p. Also,

$$(e^{t\Delta}u)(x) = \int_\Omega e^{t\Delta}\delta_x(y)u(y)\,dy. \tag{2.2}$$

The sequence of nonnegative functions $\{u_n\}$ is clearly increasing.

PROPOSITION 2.1. (i) *Suppose there is an* $(x_0, t_0) \in \Omega \times]0, T[$ *such that* $\lim_{n\to\infty} u_n(x_0, t_0) < \infty$. *Then* (P) *has a nonnegative solution on* $\Omega \times]0, T_0[$ *for all* $T_0 \in]0, t_0[$. *This solution is given by*

$$u(x, t) = \lim_{n\to\infty} u_n(x, t) \tag{2.3}$$

a.e. in $\Omega \times]0, T_0[$.

(ii) *If* (P) *has a nonnegative solution in* $\Omega \times]0, T[$, *then*

$$\lim_{n\to\infty} u_n(x, t) < \infty$$

a.e. in $\Omega \times]0, T[$.

Thus the existence of a nonnegative solution of (P) depends only on the limiting behavior of the solution of (P_n). The borderline case concerns the potential given by

$$V_0(x) = \begin{cases} c/|x|^2 & \text{if } x \in B_1, \\ 0 & \text{if } x \in \Omega\setminus B_1. \end{cases}$$

Let

$$C_*(N) = ((N-2)/2)^2.$$

THEOREM 2.2. (i) *Let* $0 \leqslant c \leqslant C_*(N)$ *and let the* (*measurable*) *potential* $V \geqslant 0$ *satisfy* $V \in L^\infty(\Omega \setminus B_1)$. *If* $V \leqslant V_0$ *in* B_1, *then* (P) *has a solution u if*

$$\int_\Omega |x|^{-\alpha} u_0 < \infty, \quad \int_0^T \int_\Omega f(x, s) |x|^{-\alpha}\, dx\, ds < \infty \tag{2.4}$$

where α *is the smallest root of* $(N - 2 - \alpha)\alpha = c$. *If* $V \geqslant V_0$ *in* B_1 *and if* (P) *has a solution u, then*

$$\int_{\Omega'} |x|^{-\alpha} u_0 < \infty, \quad \int_0^{T-\varepsilon} \int_{\Omega'} f(x, s) |x|^{-\alpha}\, dx\, ds < \infty$$

for each $\varepsilon \in \,]0, T[$ *and each* $\Omega' \subset\subset \Omega$, *where* α *is as above. If either* $u_0 \not\equiv 0$ *or* $f \not\equiv 0$ *in* $\Omega \times \,]0, \varepsilon[$ *for each* $\varepsilon \in \,]0, T[$, *then given* $\Omega' \subset\subset \Omega$ *there is a constant* $C = C(\varepsilon, \Omega') > 0$ *such that*

$$u(x, t) \geqslant C/|x|^\alpha \quad \text{if } (x, t) \in \Omega' \times [\varepsilon, T[. \tag{2.5}$$

(ii) *If* $c > C_*(N)$, $V \geqslant V_0$ *and either* $u_0 \not\equiv 0$ *or* $f \not\equiv 0$, *then* (P) *does not have a solution*.

Note that if we set $\phi(x) = |x|^{-\alpha}$, then

$$\Delta\phi = \phi_{rr} + (N - 1)\phi_r/r = \alpha(\alpha + 2 - N)|x|^{-\alpha-2},$$

so that $-\Delta\phi/\phi = c/|x|^2$, where $c = (N - 2 - \alpha)\alpha$ is as in the above theorem. Theorem 2.2 can be extended to cover potentials of the form $V = -\Delta\phi/\phi$ where $\phi > 0$, $\Delta\phi \in L^1_{\text{loc}}(\Omega)$, and a supplementary hypothesis is made (cf. Remark 7.3). In this case the conditions on the data should be

$$\int_\Omega \phi u_0 < \infty, \qquad \int_0^T \int_\Omega f(x, s)\phi(x)\, dx\, ds < \infty.$$

When $\phi(x) = |x|^{-\alpha}$, the condition $\Delta\phi \in L^1(B_1)$ becomes $N - 2 - \alpha > 0$, which holds if $c > 0$ (and $\alpha > 0$).

We finally note that the smaller root α of $(N - 2 - \alpha)\alpha = c$ is given by

$$\alpha = \frac{N-2}{2} - \left(\left(\frac{N-2}{2}\right)^2 - c\right)^{1/2}$$

when $0 \leqslant c \leqslant C_*(N)$. This is one of the technical ways in which $C_*(N)$ arises.

3. Proof of Proposition 2.1. We begin with the proof of Proposition 2.1. Part (ii) is easy. If u is a nonnegative solution of (P), then $u_n \leqslant u$ holds for all n, whence

$$\lim_{n \to \infty} u_n(x, t) \leqslant u(x, t)$$

a.e. in $\Omega \times \,]0, T[$.

Part (i) is more difficult. To start the proof, let $v_n = e^t u_n$. Then

$$\partial v_n/\partial t - \Delta v_n = (V_n + 1)v_n + e^t f_n.$$

Applying (2.1) and (2.2) to v_n gives

$$e^{t_0} u_n(x_0, t_0) \geqslant \int_0^{t_0} \int_\Omega \left(e^{(t_0 - s)\Delta}\delta_{x_0}\right)(y)(V_n(y) + 1)u_n(y, s)e^s\, dy\, ds. \tag{3.1}$$

If $\Omega' \subset\subset \Omega$ and $0 < \varepsilon < T$,

$$\inf\left\{\left(e^{s\Delta}\delta_{x_0}\right)(y)\colon (y, s) \in \Omega' \times [\varepsilon, T]\right\} = c_0 > 0.$$

Therefore

$$c_0 \int_0^{t_0-\varepsilon} \int_{\Omega'} V_n(y) u_n(y, s)\, dy\, ds + c_0 \int_0^{t_0-\varepsilon} \int_{\Omega'} u_n(y, s)\, dy\, ds \leqslant e^{t_0} u_n(x_0, t_0). \tag{3.2}$$

By hypothesis (cf. (2.3)), u_n increases and the right-hand side of (3.2) is bounded, thus u_n increases to u and $V_n u_n$ increases to Vu in $L^1(\Omega' \times]0, t_0 - \varepsilon[)$, and u is a solution of (P) in the sense of distributions. □

REMARK. This solution u satisfies the integral equation

$$u(x, t) = \int_\Omega e^{t\Delta}\delta_x(y) u_0(y) + \int_0^t \int_\Omega e^{(t-s)\Delta}\delta_x(y) V(y) u(y, s)\, dy\, ds$$
$$+ \int_0^t \int_\Omega e^{t\Delta}\delta_x(y) f(y, s)\, dy\, ds$$

a.e. in $\Omega \times]0, t_0[$. By (3.1),

$$(y, s) \mapsto e^{(t_0-s)\Delta}\delta_x(y) V(y) u(y, s) \in L^1(\Omega \times]0, t_0[)$$

since $\lim_{n\to\infty} u_n(x, t) = u(x, t) < \infty$ a.e. in $\Omega \times]0, t_0[$.

4. Proof of Theorem 2.2(i). We first show that assumption (2.4) on the data implies the existence of a solution.

Let $\phi(x) = |x|^{-\alpha}$ and let $p \in C^2(\mathbf{R})$ be a convex function satisfying $p(0) = p'(0) = 0$. Multiply the equation (cf. (P_n)) satisfied by u_n by $p'(u_n)\phi$ and integrate over $\Omega \times [\delta, t[$ for $0 < \delta < t < T$. One gets, using integration by parts,

$$\int_\Omega p(u_n(t))\phi + \int_\delta^t \int_\Omega \nabla u_n \cdot \nabla(p'(u_n)\phi) = \int_\delta^t \int_\Omega (V_n u_n + f) p'(u_n)\phi + \int_\Omega p(u_n(\delta))\phi,$$

whence, since p is convex,

$$\int_\Omega p(u_n)\phi + \int_\delta^t \int_\Omega p(u_n)(-\Delta\phi) \leqslant \int_\delta^t \int_\Omega (V_n u_n + f) p'(u_n)\phi + \int_\Omega p(u_n(\delta))\phi.$$

Replace $p(r)$ by a sequence $p_m(r)$ satisfying the hypotheses for p and converging to $|r|$ as $m \to \infty$. We obtain the limiting inequaltiy

$$\int_\Omega u_n(t)\phi + \int_\delta^t \int_\Omega u_n(-\Delta\phi) \leqslant \int_\delta^t \int_\Omega (V_n u_n + f)\phi + \int_\Omega (u_n(\delta))\phi. \tag{4.1}$$

We want to let $\delta \to 0$. First we claim that

$$\int_\Omega u_n(\delta)\phi \to \int_\Omega \phi u_0.$$

To see why this is so, note that

$$e^{\delta\Delta} u_0 \leqslant u_n(\delta) = e^{\delta(\Delta + V_n)} u_0 + \int_0^\delta e^{(\delta-s)(\Delta+V_n)} f_n(s)\, ds$$
$$\leqslant e^{\delta\lambda} e^{\delta\Delta} u_0 + \int_0^\delta e^{\lambda(\delta-s)\Delta} f_n(s)\, ds$$

if $\|V_n\|_\infty \leqslant \lambda$, since

$$e^{\delta(\Delta+V_n)}v_0 = \lim_{m\to\infty} \left(e^{\delta\Delta/m}e^{(\delta/m)V_n}\right)^m V_0 \leqslant e^{\delta\lambda}e^{\delta\Delta}V_0$$

by the positivity preserving property of $\{e^{\delta\Delta}\}$. Thus

$$\int_\Omega \left(e^{\delta\Delta}u_0\right)\phi \leqslant \int_\Omega u_n(\delta)\phi \leqslant e^{\delta\lambda}\int_\Omega \left(e^{\delta\Delta}u_0\right)\phi + e^{\delta\lambda}\delta\|f_n\|_\infty\left(\int_\Omega \phi\right),$$

whence

$$\int_\Omega \left(e^{\delta\Delta}u_0\right)\phi = \int_\Omega \left(e^{\delta\Delta}\phi\right)u_0 \to \int_\Omega \phi u_0$$

as $\delta \to 0$, as asserted. Letting $\delta \to 0$ in (4.1), we deduce

$$\int_\Omega u_n(t)\phi + \int_0^t\int_\Omega u_n(-\Delta\phi) \leqslant \int_0^t\int_\Omega V_n u_n\phi + \int_0^t\int_\Omega f_n\phi + \int_\Omega \phi u_0.$$

But

$$-\Delta\phi = (N-2-\alpha)\alpha/|x|^{\alpha+2} \geqslant V_n\phi$$

since $c = (N-2-\alpha)\alpha$. Consequently

$$\int_\Omega u_n(t)\phi \leqslant \int_0^t\int_\Omega f_n\phi + \int_\Omega \phi u_0, \tag{4.2}$$

and therefore if

$$\int_0^t\int_\Omega f_n\phi + \int_\Omega \phi u_0 < \infty$$

we conclude that $u_n(x,t)$ increases to a finite limit $u(x,t)$ as $n \to \infty$, for all $t \in]0,T]$ and for a.e. $x \in \Omega$. By Proposition 2.1, this proves the first part of Theorem 2.2(i).

Thus the problem (P) has a solution for $u_0 = \phi(x)^{-1}\delta_x$ and $f \equiv 0$, where $x \in \Omega\backslash\{0\}$ is fixed. Let u_x denote this solution and let $h_x(y,t) = u_x(y,t)/\phi(y)$. Let also $h = u/\phi$, $h_n = u_n/\phi$ where u and u_n are the solutions of (P) and (P_n) constructed above. We claim that

$$h(x,t) = \int_\Omega h_x(y,t)\phi(y)u_0(y) + \int_0^t\int_\Omega h_x(y,t-s)f(y,s)\phi(y). \tag{4.3}$$

For the proof, let u_n and v_n satisfy

$$\partial u_n/\partial t - \Delta u_n = V_n u_n + f_n, \qquad u_n = 0 \quad \text{on } \partial\Omega, \qquad u_n(0) = u_0;$$
$$\partial v_n/\partial t - \Delta v_n = V_n v_n, \qquad v_n = 0 \quad \text{on } \partial\Omega, \qquad v_n(0) = \phi(x)^{-1}\delta_x.$$

Then

$$\frac{\partial}{\partial s}\int_\Omega u_n(s)v_n(t-s)\,dx = \int_\Omega f_n(s)v_n(t-s)\,dx,$$

whence

$$\int_\Omega u_n(t-\delta)v_n(\delta)\,dx = \int_\Omega u_n(t-\delta)v_n(\delta)\,dx + \int_\delta^{t-\delta}\int_\Omega f_n(s)v_n(t-s)\,dx\,ds. \tag{4.4}$$

As $\delta \to 0$,

$$u_n(t-\delta) \to u_n(t) \quad \text{and} \quad v_n(t-\delta) \to v_n(t) \quad \text{in } C(\Omega),$$
$$v_n(\delta) \rightharpoonup \phi(x)^{-1}\delta_x \quad \text{and} \quad u_n(\delta) \rightharpoonup u_0$$

where $\rightharpoonup$ denotes weak convergence. Thus when $\delta \to 0$ we obtain, from (4.4),

$$u_n(x,t)\phi(x)^{-1} = \int_\Omega v_n(y,t)u_0(y) + \int_0^t \int_\Omega f_n(y,s)v_n(y,t-s). \tag{4.5}$$

When $n \to \infty$,

$$u_n(x,t) \uparrow u(x,t) = h(x,t)\phi(x), \qquad v_n(y,t) \uparrow h_x(y,t)\phi(y),$$

and taking the limit in (4.5) gives (4.3).

Our next assertion is that if $V \geqslant V_0$ and $u_0 \not\equiv 0$, for $\varepsilon > 0$ and $\Omega' \subset\subset \Omega$ with $0 \in \Omega'$, there is a $C > 0$ such that

$$h(x,t) \geqslant C \tag{4.6}$$

for all $x \in \Omega'$ and $t \in [\varepsilon, T[$. For the proof we first recall that if $u_0 \not\equiv 0$, there is a positive constant C_0 such that $e^{t\Delta}u_0(y) \geqslant C_0$ if $x \in \Omega'$ and $t \in [\varepsilon/2, T[$. Next u is bounded below by the solution w of

$$\partial w/\partial t - \Delta w = V_0 w \quad \text{in } \mathcal{D}'(\Omega \times [\varepsilon/2, T[),$$
$$w = 0 \quad \text{on } \partial\Omega, \qquad w(y, \varepsilon/2) = C_0\chi_{\Omega'}(y) \quad \text{in } \Omega,$$

and w is the (increasing) limit of the unique nonnegative solution w_n of

$$\partial w_n/\partial t - \Delta w_n = V_n w_n \quad \text{in } \mathcal{D}'(\Omega \times]\varepsilon/2, T[),$$
$$w_n = 0 \quad \text{on } \partial\Omega, \qquad w_n(y, \varepsilon/2) = C_0\chi_{\Omega'}(y) \quad \text{in } \Omega.$$

Choose a ball B in Ω', centered at the origin and of radius r_0. Then $w_n \geqslant v_n$ where

$$\begin{aligned} &\partial v_n/\partial t - \Delta v_n = V_n v_n \quad \text{in } \mathcal{D}'(B \times]\varepsilon/2, T[),\\ &v_n = 0 \quad \text{on } \partial B, \qquad v_n(y, \varepsilon/2) = C_0 \quad \text{in } B, \end{aligned} \tag{4.7}$$

where here and in the sequel, $V_n = \inf(V_0, n)$. But v_n is a radial function, i.e. a function of $|x| = r$ alone. Thus

$$\begin{aligned} &\frac{\partial v_n}{\partial t} - \frac{\partial^2 v_n}{\partial r^2} - \frac{(N-1)}{r}\frac{\partial v_n}{\partial r} = V_n v_n,\\ &v_n(r_0, t) = 0 = \frac{\partial v_n}{\partial r}(0,t), \qquad v_n\left(r, \frac{\varepsilon}{2}\right) = C_0. \end{aligned} \tag{4.8}$$

Multiply (4.7) by $(v_n)^{p-1}\phi^{2-p}$ for $p > 1$ and integrate to obtain

$$\frac{\partial}{\partial t}\left(p^{-1}\int_B \left(\frac{v_n}{\phi}\right)^p \phi^2\right) + \int_B \nabla v_n \cdot \nabla\left(v_n^{p-1}\phi^{2-p}\right) = \int_B V_n (v_n/\phi)^p \phi^2.$$

Setting $k_n = v_n/\phi$ we get

$$\frac{\partial}{\partial t}\left(p^{-1}\int_B k_n^p \phi^2\right) + 4(p-1)p^{-2}\int_B |\nabla k_n^{p/2}|^2\phi^2 + \int_B k_n^p(-\Delta\phi)\phi = \int_B V_n k_n^p \phi^2.$$

Recall (cf. §2) that $V_n(x) \leqslant V_0(x) \leqslant -\Delta\phi/\phi$. Thus $V_n\phi^2 \leqslant (-\Delta\phi)\phi$ and consequently

$$\frac{\partial}{\partial t}\left(p^{-1}\int_B k_n^p\phi^2\right) \leqslant 0,$$

whence for $\varepsilon/2 \leqslant t < T$,

$$\left(\int_B v_n^p\phi^{2-p}\right)^{1/p}(t) \leqslant C_0\left(\int_B \phi^{2-p}\right)^{1/p},$$

the right side being the value of the left side for $t = \varepsilon/2$. Letting $p \to \infty$ it follows that $k_n \leqslant C_0$ a.e. in B, which is equivalent to $v_n \leqslant C_0\phi$ a.e. in B. We are now justified in setting

$$v = \lim_{n\to\infty} v_n, \qquad k = \lim_{n\to\infty} k_n.$$

We will show that

$$C_0 \geqslant k(x,t) \geqslant C_1 > 0 \quad \text{for } \varepsilon < t < T \text{ and a.e. } x \in \tfrac{1}{2}B. \tag{4.9}$$

(Here $B = B_{r_0}$, $\frac{1}{2}B = B_{r_0/2}$ and $k \leqslant C_0$ is already proven.) Since

$$u \geqslant \phi h \geqslant w \geqslant w_n \geqslant v_n \geqslant k_n\phi,$$

(4.9) implies (4.6) with $y \in \Omega' = \frac{1}{2}B$. And for $y \in \Omega' \setminus (\frac{1}{2}B)$ we have (since $u \geqslant e^{t\Delta}u_0$)

$$h(y,t) \geqslant \phi(y)^{-1}(e^{t\Delta}u_0)(y) \geqslant C_2 > 0$$

for all $y \in \Omega'$,

$$\phi(y)^{-1} \geqslant C_3 > 0 \quad \text{in } \Omega \setminus \tfrac{1}{2}B,$$

where C_2 and C_3 are suitable constants. We now establish (4.9), using ideas of J. Moser [**4, 5**].

Let $g\colon [0,\infty[\to [0,\infty[$ be convex and of class C^2. Multiply (4.7) by $g'(k_n)g(k_n)\phi\psi^2$ where $\psi \in \mathfrak{D}(B \times]\varepsilon/2, T[)$ and integrate over $Q = B \times]\varepsilon/2, T[$. We obtain

$$\int_Q \frac{1}{2}\left(\frac{\partial}{\partial t}\left(g(k_n)^2\right)\right)\phi^2\psi^2 + \int_Q \nabla(k_n\phi)\cdot\nabla\left(g'(k_n)g(k_n)\phi\psi^2\right)$$
$$= \int_Q V_n k_n \phi g'(k_n)g(k_n)\psi^2\phi.$$

Straightforward computations give

$$\begin{aligned}
&\int_B \nabla(k_n\phi)\cdot\nabla\left(g'(k_n)g(k_n)\phi\psi^2\right)\\
&\quad= \int_B \nabla(k_n)\cdot\nabla\left(g'(k_n)g(k_n)\phi\psi^2\right)\phi + \int_B k_n\nabla\phi\cdot\nabla\left(g'(k_n)g(k_n)\phi\psi^2\right)\\
&\quad= \int_B \left(g''(k_n)|\nabla k_n|^2 g(k_n)\phi^2\psi^2\right) + \int_B |\nabla g(k_n)|^2\phi^2\psi^2\\
&\qquad+ \int_B (\nabla g(k_n)\cdot\nabla\phi)\phi\psi^2 g(k_n) + \int_B \nabla g(k_n)\cdot(\nabla\psi^2)g(k_n)\phi^2\\
&\qquad+ \int_B (-\Delta\phi)k_n g'(k_n)g(k_n)\phi\psi^2 - \int_B (\nabla\phi\cdot\nabla g(k_n))g(k_n)\phi\psi^2,
\end{aligned}$$

whence

$$\int_Q \frac{1}{2}\left(\frac{\partial}{\partial t}\left(g(k_n)^2\right)\right)\phi^2\psi^2 + \int_Q |\nabla g(k_n)|^2\phi^2\psi^2$$
$$+\int_Q g''(k_n)|\nabla k_n|^2\left(g(k_n)\phi^2\psi^2\right) + \int_Q\left(\nabla g(k_n)\cdot\nabla\psi^2\right)g(k_n)\phi^2$$
$$+\int_Q(-\Delta\phi)k_n g'(k_n)g(k_n)\phi\psi^2 = \int_Q V_n k_n\phi^2 g'(k_n)g(k_n)\psi^2.$$

The third term on the left is nonnegative since g is convex and nonnegative; we will integrate the first term by parts and for the fourth term use the trivial inequality

$$\left|2\int_B(\nabla g(k_n)\cdot\nabla\psi)g(k_n)\phi^2\psi\right| \leqslant \frac{1}{2}\int_B|\nabla g(k_n)|^2\phi^2\psi^2 + 2\int_B|\nabla\psi|^2 g|k_n|^2\phi^2.$$

We thus obtain

$$2^{-1}\left(\int_B g(k_n)^2\psi^2\phi^2\right)(t) + 2^{-1}\int_Q|\nabla g(k_n)|^2\phi^2\psi^2 \leqslant \int_Q\phi(V_n\phi+\Delta\phi)k_n g'(k_n)g(k_n)\psi^2$$
$$+\int_Q g(k_n)^2\phi^2\left(2|\nabla\psi|^2 + \psi\frac{\partial\psi}{\partial t}\right).$$

Now suppose $\phi\,\triangle\,\phi\in L^1(B)$ (which is equivalent to $\alpha<(N-2)/2$). Take B to have sufficient by small radius. Since $V_0 = -\Delta\phi/\phi$ the first term on the right side of the above inequality tends to zero as $n\to\infty$ by Lebesgue's dominated convergence theorem. (Here we are using $\|k_n\|_\infty \leqslant C_0$ in B and the hypotheses on g.) Thus when $n\to\infty$ we obtain
(4.10)

$$\int_B g(k(t))^2\psi(t)^2\phi^2 + \int_Q|\nabla(g(k))|^2\phi^2\psi^2 \leqslant 2\int_Q g(k)^2\phi^2\left(2|\nabla\psi|^2+\psi\frac{\partial\psi}{\partial t}\right).$$

Now choose ψ so that $0\leqslant\psi\leqslant 1$, $\psi=1$ in $B_{r-\delta}\times[s+\delta,T]$, $\psi=0$ in $(B\times[0,s])\cup((B\setminus B_r)\times[0,T])$ where $0<s,\delta$. We further suppose that

$$|\partial\psi/\partial t|\leqslant C_4/\delta, \qquad |\nabla\psi|^2\leqslant C_4/\delta^2$$

where the constant C_4 is independent of the pair (s,δ). Inequality (4.10) then yields

$$(4.11)\qquad \int_{B_{r-\delta}} g(k(t))^2\phi^2 + \int_{s+\delta}^T\int_{B_{r-\delta}}|\nabla g(k)|^2\phi^2 \leqslant 6C_4\delta^{-2}\int_s^T\int_{B_r} g(k)^2\phi^2$$

for all $t\in[s+\delta,T[$.

If $0<r'\leqslant r\leqslant 1$ and h is a nice function on $[0,r]$, then

$$(4.12)\qquad \left(\int_0^r|h(s)|^m s^{N-2\alpha-1}\,ds\right)^{2/m} \leqslant C_5\left(\int_0^r\left[|h'(s)|^2+|h(s)|^2\right]s^{N-2\alpha-2}\,ds\right)$$

where $1/m\geqslant 1/2-1/(N-2\alpha)$ (and $m<\infty$ if $N-2\alpha=2$). The constant C_5 depends on r' but not on r.

REMARK 4.1. If $M > 2$ is given and if

$$2 \leqslant m \leqslant \min\{M, 2(N-2\alpha)(N-2\alpha-2)^{-1}\},$$

then the constant C_5 in (4.12) is uniformly bounded for $\alpha \in [0, (N-2)/2]$.

A proof of (4.12) which contains a description of the constant C_5 is given in the Appendix.

Define β by $\beta + 2/m = 1$ where $1 > 2/m \geqslant 1/2 - 1/(N-2\alpha)$. By Hölder's inequality and (4.12) we get, for a nonnegative radial function h,

$$\begin{aligned}\int_{B_r} h^{2+2\beta}\phi^2 &\leqslant \left(\int_{B_r} h^m\phi^2\right)^{2/m}\left(\int_{B_r} h^2\phi^2\right)^{\beta} \\ &\leqslant C_5\left(\int_{B_r} |\nabla h|^2\phi^2 + \int_{B_r} h^2\phi^2\right)\left(\int_{B_r} h^2\phi^2\right)^{\beta}\end{aligned}$$

whence

$$\int_a^b\int_{B_r} h^{2+2\beta}\phi^2 \leqslant C_5\left(\int_a^b\int_{B_r}(|\nabla h|^2 + h^2)\phi^2\right)\sup_{a\leqslant t\leqslant b}\left(\int_{B_r} h^2\phi^2\right)^{\beta}(t). \tag{4.13}$$

From (4.11) we deduce

$$\sup_{t\in[s+\delta, T]}\int_{B_{r-\delta}} g(k(t))^2\phi^2 \leqslant 6C_4\delta^{-2}\int_s^T\int_{B_r} g(k)^2\phi^2.$$

Note that $x \mapsto g(k(x,t))$ is, for fixed t, a radial function, so applying (4.13) with $[a,b] = [s+\delta, T]$ and with $B_{r-\delta}$ in place of B_r we get

$$\begin{aligned}\int_{s+\delta}^T\int_{B_{r-\delta}} g(k)^{2+2\beta}\phi^2 &\leqslant C_5(6C_4\delta^{-2}+1)\left(\int_s^T\int_{B_r} g(k)^2\phi^2\right) \\ &\quad\cdot\left(6C_4\delta^{-2}\int_s^T\int_{B_r} g(k)^2\phi^2\right)^{\beta},\end{aligned}$$

whence

(4.14)

$$\begin{aligned}\left(\int_{s+\delta}^T\int_{B_{r-\delta}} g(k)^{2+2\beta}\phi^2\right)^{1/(2+2\beta)} &\leqslant \left[C_5^{1/2}(6C_4+1)\right]^{1/(1+\beta)}\delta^{-\gamma}\left(\int_s^T\int_{B_r} g(k)^2\phi^2\right)^{1/2} \\ &= C_6\delta^{-1}\left(\int_s^T\int_{B_r} g(k)^2\phi^2\right)^{1/2}\end{aligned}$$

since the exponent of δ is $-\gamma = -(1+\beta)^{-1}(1+\beta) = -1$ and $0 < \delta \leqslant 1$.

Let $a > 0$ be a small number and let

$$\delta = a/2^n, \qquad r_{n+1} = r_n - a/2^n, \qquad g_{n+1} = g_n^{1+\beta}, \qquad s_{n+1} = s_n + a/2^n,$$

$$H_n = \left(\int_{s_n}^T\int_{B_{r_n}} g_n(k)^2\phi^2\right)^{1/2}$$

where $g_1 = g$, and r_1 and s_1 are given positive numbers. With this notation the estimate (4.14) yields

$$H_{n+1}^{1/(1+\beta)} \leqslant C_6 \cdot 2^n a^{-1} H_n,$$

whence, by induction,

$$H_n^{1/(1+\beta)} \leqslant \left(C_6 a^{-1}\right)^{\alpha_n} 2^{\gamma_n} H_1^{(1+\beta)^{n-2}}$$

where

$$\alpha_n = (1+\beta)^{n-2} \sum_{j=0}^{n-2} (1+\beta)^{-j}, \qquad \gamma_n = \sum_{j=0}^{n-1} (j+1)(1+\beta)^{n-2-j}.$$

Now let $n \to \infty$. Since $g_n = g^{(1+\beta)^{n-1}}$ we get

$$\sup_{B_{r_1-a}\times[s_1+a,T]} g(k(x,t)) \leqslant \left(C_6 a^{-1} \cdot 2^{(1+\beta)/\beta}\right)^{(1+\beta)/\beta} \left(\int_{s_1}^{T}\int_{B_{r_1}} g(k)^2 \phi^2\right)^{1/2}.$$

Replace g by a sequence $\{g_l\}$ satisfying the hypotheses and tending to $1/r^\gamma$ as $l \to \infty$. We then obtain

$$\sup_{B_{r_1-a}\times[s_1+a,T]} 1/k^\gamma \leqslant \left(C_6 a^{-1} \cdot 2^{(1+\beta)/\beta}\right)^{(1+\beta)/\beta} \left(\int_{s_1}^{T}\int_{B_{r_1}} k^{-2\gamma}\phi^2\right)^{1/2}.$$

Now set

$$s_1 = 3\varepsilon/4, \qquad a = \varepsilon/4, \qquad r_1 < r_0$$

where $\varepsilon > 0$ is given (and r_0 is as before, cf. the sentence preceding (4.7)). Note that

$$k(x,t) = v/\phi \geqslant \phi|x|^{-1}\left(e^{t\Delta}v_0\right)(x) \geqslant C_0 C_7 \phi(x)^{-1}$$

for $(x,t) \in B_{r_1} \times]3\varepsilon/4, T[$ where the constant C_7 is independent of r_1 and ε (but, of course, C_0 depends on ε, as before). Thus we obtain

$$\sup_{B_{r_1-\varepsilon/4}\times[\varepsilon,T]} k^{-\gamma} \leqslant C_8 C_0^{-\gamma}\varepsilon^{-1-1/\beta}\left(\int_{3\varepsilon/4}^{T}\int_{B_{r_1}} \phi^{2+2\gamma}\right)^{1/2},$$

which implies the estimate

$$k(x,t) \geqslant C_9 C_0 \varepsilon^{(1+1/\beta)(1/\gamma)}\left(\int_{B_{r_1}} \phi^{2+2\gamma}\right)^{-1/2\gamma} \tag{4.15}$$

for a.e. $x \in B_{r_1-\varepsilon/4}$ and for all $t \in [\varepsilon, T]$. Here the positive constant C_9 is independent of the pair (r_1, ε). The estimate (4.9) is therefore established for $\alpha < (N-2)/2$, and it also holds for the limiting case of $\alpha = (N-2)/2$.

[First of all, it is easy to see that $\gamma > 1$ can be chosen so that $\int_{B_{r_1}} \phi^{2+2\gamma} < \infty$ if $\alpha \leqslant (N-2)/2$. In fact, $\gamma = 1 + 3/(N-2)$ will do (for $N \geqslant 3$). Next, the mapping $\alpha \to k(x,t)$ is continuous. To see why, note that the solution u_α of (P) with $V(x) = (N-2-\alpha)\alpha/|x|^2$ depends continuously (and monotonically) on α for $0 \leqslant \alpha \leqslant (N-2)/2$. Therefore $\alpha \to k_\alpha |x|^{-\alpha}$ is monotonically increasing, and so

$$k_{(N-2)/2}|x|^{-(N-2)/2} \geqslant k_\alpha |x|^{-\alpha}.$$

Remark 4.1 implies that for $\beta = 1 - 1/2m$ with $m = \min\{M, 2(N-2\alpha)/(N-2\alpha-2)\}$, the constant C_9 in (4.15) is bounded below by a positive constant independent of $\alpha \in [0, (N-2)/2]$. Thus there is a $C > 0$ such that $k_\alpha |x|^{-\alpha} \geq C|x|^{-\alpha}$ for $x \in \frac{1}{2}B$, $t \geq \varepsilon$ and $\alpha \in [0, (N-2)/2]$. Hence

$$k_{(N-2)/2} \geq C|x|^{(N-2)/2-\alpha}$$

where C is independent of α. A passage to the limit now gives the result.]

The estimate (2.5) is an immediate consequence of (4.6). To establish the rest of Theorem 2.2 (i), apply (4.6) with $u_0 = \delta_x$; we obtain the existence of a constant $C > 0$ such that

$$h_x(y,t) \geq C/\phi(x) \quad \text{for all } (x,y) \in \Omega' \times \Omega', t \in [\varepsilon, T].$$

Then (4.3) implies

$$\phi(x)h(x,t) \geq C\int_{\Omega'} \phi u_0 + C\int_0^{T-\varepsilon}\int_{\Omega'} f\phi. \tag{4.16}$$

If a solution u exists then necessarily $h(x,t) < \infty$ for a.e. $(x,t) \in \Omega \times]0,T]$. Consequently the condition

$$\int_{\Omega'} \phi u_0 < \infty, \quad \int_0^{T-\varepsilon}\int_{\Omega'} f\phi < \infty$$

are necessary for the existence of a solution u. □

5. Proof of Theorem 2.2(ii). In this section we deduce it from the first part of the theorem. An independent proof based on probability theory will be given in the next section.

Let $c > C_*(N)$. If (P) has a solution $u \not\equiv 0$, then one has

$$\partial u/\partial t - \Delta u = C_*(N)|x|^{-2}u + \left(c - C_*(N)\right)|x|^{-2}u$$

in $\mathcal{D}'(\Omega \times]0,T[)$. From part (i) we know that the solution exists only if

$$\left[c - C_*(N)\right]u|x|^{-2}\phi \in L^1(\Omega' \times]0, T-\varepsilon[)$$

for $\Omega' \subset\subset \Omega$ and $\varepsilon > 0$ (where we assume $0 \in \Omega'$). By part (i) (see (2.5)) we have $u \geq C_\varepsilon |x|^{-C_*(N)}$ in $\Omega' \times [\varepsilon, T[$, whence $|x|^{-2-2C_*(N)} \in L^1(\Omega')$, which is false. □

While this is a short proof it is not a simple one in view of the estimates that went into the proof of Theorem 2.2(i).

6. The Feynman-Kac formula applied to nonexistence. We consider the problem

$$\text{(P')} \quad \begin{aligned} &\partial u/\partial t = \tfrac{1}{2}\Delta u + V(x)u && (x \in \mathbf{R}^N, t > 0), \\ &u(x,0) = u_0(x) && (x \in \mathbf{R}^N), \end{aligned}$$

where, as before, $V \in L^\infty_{\text{loc}}(\mathbf{R}^N \setminus \{0\})$. The normalization factor $\frac{1}{2}$ has been introduced for the following reason: If V is bounded above, then the unique nonnegative solution of (P′) is given by

$$u(x,t) = \int_S \exp\left(\int_0^t V(\omega(s))\,ds\right) u_0(\omega(t)) P_x(d\omega)$$

where ω is in the space of paths $S = C([0, \infty[; \mathbf{R}^N)$ and where P_x is the Wiener probability measure starting at x, i.e. P_x is supported on $\{\omega \in S: \omega(0) = x\}$.

THEOREM 6.1. *Suppose u_0 is measurable, $u_0 \geqslant 0$ and $u_0 \not\equiv 0$. Suppose further that $V(x) \geqslant \nu(|x|)$ where ν is a nonnegative measurable function on $[0, \infty[$ satisfying*

$$\liminf_{r \to 0^+} r^2 \nu(r) > \frac{N^2 \pi^2}{8}. \tag{6.1}$$

Let (P'_n) be the problem (P') but with V replaced by $V_n = \min\{n, V\}$, and let u_n be the unique nonnegative solution of (P'_n). Then $\lim_{n \to \infty} u_n(x, t) = \infty$ for each $(x, t) \in \mathbf{R}^N \times]0, \infty[$.

We shall give a direct probabilistic proof of this, independent of the previous sections. Before doing so we make some remarks. The change of variables $x \mapsto \sqrt{2}\, x$ transforms the equation $\partial u/\partial t = \frac{1}{2}\Delta u + V_n$ into $\partial u/\partial t = \Delta u + \tilde{V}u$ where $\tilde{V}(x) = V(x/\sqrt{2})$. Thus if $V(x) \geqslant c/|x|^2$, (6.1) implies that $c > N^2\pi^2/4$ is sufficient condition for nonexistence. Of course, by Theorem 2.2, $c > C_*(N)$ is the optimal sufficient condition and $C_*(N) < N^2\pi^2/4$. The probabilistic proof given below by itself does not appear to be capable of giving the sharpest result. (But see the remark at the end of this section.) Probabilistic arguments can extend the result of Theorem 6.1 from the whole space case ($x \in \mathbf{R}^N$) to the bounded domain case ($x \in \Omega \subset\subset \mathbf{R}^N$ with the Dirichlet condition $u = 0$ on $\partial\Omega$ imposed). We omit further discussion of this.

For the proof of the theorem, fix $t > 0$, $x \in \mathbf{R}^N$. Since

$$u_n(x, t) = \int_S \exp\left(\int_0^t V_n(\omega(s))\, ds\right) u_0(\omega(t)) P_x(d\omega)$$

it follows that

$$u_n(x, t) \uparrow u(x, t) = \int_S \exp\left(\int_0^t V(\omega(s))\, ds\right) u_0(\omega(t)) P_x(d\omega)$$

by Lebesgue's monotone convergence theorem. What we must prove is that $u(x, t) = \infty$.

To simplify the proof somewhat we assume u_0 is strictly positive in some ball, i.e.

$$u_0(x) \geqslant \varepsilon_0 \quad \text{for } |x - x_0| \leqslant \delta_0 \tag{6.2}$$

for some choice of $\varepsilon_0 > 0$, $\delta_0 > 0$, $x_0 \in \mathbf{R}^N$. This can be assumed without loss of generality since for $0 < \varepsilon < t$, $u_n(x, t)$ $(\leqslant u(x, t))$ is the solution at $(x, t - \varepsilon)$ of (P'_n) having initial data $u_n(x, \varepsilon)$ which is everywhere continuous and positive.

Let $0 < \alpha < \frac{1}{2}$ and let

$$S_n = \{\omega \in S: \omega(0) = x, |\omega(t) - x_0| \leqslant \delta_0, \text{ and } |\omega(s)| < 1/n \text{ for } s \in J = [\alpha t, (1 - \alpha)t]\}.$$

Our hypotheses imply

$$u(x, t) \geqslant \int_{S_n} \exp\left(\int_{\alpha t}^{(1-\alpha)t} V(\omega(s))\, ds\right) u_0(\omega(t)) P_x(d\omega)$$

$$\geqslant \exp\left(\gamma t \nu\left(\frac{1}{n}\right)\right) \varepsilon_0 P_x(S_n)$$

where $\gamma = 1 - 2\alpha$. The *main estimate* is

$$(6.3) \qquad u(x,t) \geqslant \exp(t\gamma\nu(1/n))\varepsilon_0 k_0 \exp\{-\pi^2 N^2 \gamma t n^2/(8-\varepsilon)\} n^{-N}.$$

This implies $u(x,t) = \infty$ by (6.1) since $\varepsilon > 0$ is arbitrary. The exact dependence of the constant k_0 on (γ, t, x, x_0) will be made clear in the sequel (cf. (6.11), (6.12)).

The proof of the main estimate (6.3) is based on connections between Brownian motion and the heat equation.

To simplify matters further we work in $N = 1$ space dimension. The estimate for N dimensions follows from the one-dimensional estimate (or rather its proof). The one-dimensional proof below could be done in N dimensions, but the separation of variables part of the argument is clearest when $N = 1$.

Let $\{\beta(t,\omega): t \geqslant 0, \omega \in S\}$ be normalized one-dimensional Brownian motion (cf. [3]). Let $-\infty < a < b < \infty$ and let $\tau = \tau([a,b])$ be the exit time from $[a,b]$ for Brownian motion starting at x, i.e.,

$$\tau(\omega) = \inf\{t > 0: \beta(t,\omega) = a \text{ or } b, \text{ given that } \beta(0,\omega) = x\}.$$

For typographical convenience we shall usually suppress ω. Then w, defined by

$$w(x,t) = P_x\{\omega \in \Omega: \tau(\omega) > t\} = P_x\{\tau > t\},$$

satisfies

$$(6.4) \qquad \begin{array}{ll} \partial w/\partial t = \frac{1}{2}\partial^2 w/\partial x^2 & \text{for } a < x < b,\ t > 0, \\ w(t,a) = w(t,b) = 0 & \text{for } t > 0, \\ w(0,x) = 1 & \text{for } a < x < b. \end{array}$$

For a nice discussion see [**6**, p. 207].

For $b, t_0 > 0$,

$$(6.5) \qquad \begin{aligned} P_0\{|\beta(s)| < b \text{ for } 0 \leqslant s \leqslant t_0\} &= P_0\{\tau([-b,b]) > t_0\} \\ &= P_b\{\tau([0,2b]) > t_0\} = w(b,t_0) \end{aligned}$$

where w satisfies (6.4) with $[a,b]$ replaced by $[0,2b]$. We calculate w by separation of variables, obtaining

$$w(x,t) = \sum_{n \text{ odd}} \frac{4}{\pi} \exp\left(-\frac{n^2\pi^2 t}{8b^2}\right) \sin\left(\frac{n\pi x}{2b}\right),$$

hence

$$\begin{aligned} w(b,t_0) &= \sum_{n \text{ odd}} \frac{4}{\pi} \exp\left(-\frac{n^2\pi^2 t_0}{8b^2}\right) \sin\left(\frac{n\pi}{2}\right) \\ &= \sum_{k=0}^{\infty} \frac{4}{\pi}(-1)^k \exp\left(-(2k+1)^2\frac{\pi^2 t_0}{8b^2}\right) \\ &\geqslant \frac{4}{\pi}\left\{\exp\left(-\frac{\pi^2 t_0}{8b^2}\right) - \exp\left(-\frac{9\pi^2 t_0}{8b^2}\right)\right\} \end{aligned}$$

because this alternating series exceeds the sum of its first two terms. Taking $b = 1/n$ and $t_0 = t$ gives

$$w\left(\frac{1}{n}, t\right) \geq \frac{4}{\pi}\left\{\exp\left(-\frac{\pi^2 n^2 t}{8}\right) - \exp\left(-\frac{9\pi^2 n^2 t}{8}\right)\right\}$$
$$\geq \frac{4-\varepsilon_1}{\pi}\exp\left(-\frac{\pi^2 n^2 t}{8}\right)$$

for each $\varepsilon_1 > 0$ and all $n > N_1(\varepsilon_1, t)$. Thus, for large n,

$$w(1/n, t) \geq c_1 \exp(-\pi^2 n^2 t/8) \tag{6.6}$$

where c_1 is any number less than $4/\pi$.

Let $x, x_0 \in \mathbf{R}^N$ and $t, \delta_0 > 0$ be as before. Then, using the strong Markov property repeatedly, we obtain, for $0 < \alpha < \frac{1}{2}$, $0 < \alpha_1 < 1$,

(6.7)

$$\begin{aligned}
&P_x\{|\beta(s)| < 1/n \text{ for all } s \in J = [\alpha t, (1-\alpha)t], |\beta(t) - x_0| \leq \delta_0\}\\
&\quad \geq P_x\{|\beta(\alpha t)| < \alpha_1/n, |\beta(s)| < 1/n \text{ for } s \in J, |\beta(t) - x_0| \leq \delta_0\}\\
&\quad = P_x\{|\beta(\alpha t)| < \alpha_1/n\}\\
&\qquad \cdot P\{|\beta(s)| < 1/n \text{ for } s \in J, |\beta(s) - x_0| \leq \delta_0 | \beta(\alpha t) \leq \alpha_i/n\}\\
&\quad = P_x\{|\beta(\alpha t)| < \alpha_1/n\} \cdot P\{|\beta(s)| < 1/n \text{ for } s \in J | \beta(\alpha t) \leq \alpha_1/n\}\\
&\qquad \cdot P\{|\beta(t) - x_0| \leq \delta_0 || \beta((1-\alpha)t)| < 1/n\}\\
&\quad \geq P_x\{|\beta(\alpha t)| < \alpha_1/n\} \cdot P_0\{|\beta(s)| < (1-\alpha_1)/n \text{ for } s \in [0, (1-2\alpha)t]\}\\
&\qquad \cdot P\{|\beta(\alpha t) - x_0| \leq \delta_0 || \beta(0)| < 1/n\} \equiv \rho_1\rho_2\rho_3.
\end{aligned}$$

Now, if $t_1 = \alpha t$,

$$\begin{aligned}
\rho_1 &= \frac{1}{\sqrt{2\pi t_1}}\int_{-\alpha_1/n}^{\alpha_1/n} \exp\left(\frac{(s-x)^2}{2t_1}\right) ds\\
&= \frac{1}{\sqrt{2\pi}}\int_{(-x-\alpha_1/n)/\sqrt{t_1}}^{(-x+\alpha_1/n)/\sqrt{t_1}} e^{-y^2/2}\, dy\\
&\geq \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{(|x| + \alpha_1/n)^2}{2t_1}\right\}\left(\frac{2\alpha_1}{n\sqrt{t_1}}\right)\\
&\geq (2\pi\alpha t)^{-1/2}\left\{\exp\left(\frac{-x^2}{2\alpha t}\right) - \varepsilon_2\right\} \cdot 2\alpha_1/n
\end{aligned} \tag{6.8}$$

for each $\varepsilon_2 > 0$ and all $n > N_2(\varepsilon_2, x, t, \alpha)$.

Similarly, ρ_3 is a weighted average of terms of the form

$$(2\pi t_1)^{-1/2}\int_{x_0-\delta_0}^{x_0+\delta_0} \exp\left[-\frac{(s-\beta(0))^2}{2t_1}\right] ds$$

where $t_1 = \alpha t$ and $\beta(0)$ ranges from $-1/n$ to $1/n$. Consequently,

$$\rho_3 \geqslant (2\pi\alpha t)^{-1/2}\left[\exp\left\{-\frac{(|x_0|+\delta_0)^2}{2\alpha t}\right\} - \varepsilon_3\right]\cdot 2\delta_0 \tag{6.9}$$

for each $\varepsilon_3 > 0$ and all $n > N_3(\varepsilon_3, x, t, \alpha)$. Next,

$$\begin{aligned}\rho_2 &= P_0\{|\beta(s)| < (1-\alpha_1)/n \text{ for } s \in [0, (1-2\alpha)t]\}\\ &= w((1-2\alpha)t, (1-\alpha_1)/n)\end{aligned}$$

by (6.5). Hence by (6.6),

$$\rho_2 \geqslant \frac{(4-\varepsilon_3)}{\pi}\exp\left\{-\pi^2 n^2 \frac{(1-2\alpha)t}{8(1-\alpha_1)^2}\right\} \tag{6.10}$$

for any $\varepsilon_3 > 0$ and all $n > N_3(\varepsilon_3, x, t, \alpha, \alpha_1)$. By (6.6) and (6.8)–(6.10) we see that given $\varepsilon > 0$, there is an $\varepsilon_4 < \varepsilon$ and an $N = N(\varepsilon, x, t)$ such that for $n > N$,

$$\begin{aligned}P_x(S_n) \geqslant \rho_1\rho_2\rho_3 &\geqslant (2\pi\alpha t)^{-1}\frac{4\pi\alpha_1\delta_0}{n}\exp\left\{-\left(x^2/2\alpha t + \frac{(|x_0|+\delta_0)^2}{2\alpha t} + \varepsilon_4\right)\right\}\\ &\cdot\frac{4}{\pi}\exp\left\{-\pi^2 n^2\frac{(1-2\alpha)t}{8(1-\alpha_1)^2}\right\}.\end{aligned}$$

Taking α_1 so that $(1-\alpha_1)^2 = 1-\varepsilon$ gives the estimate (6.3) with $N = 1$ where

$$k_0 = \left(\frac{8\alpha_1\delta_0}{\pi\alpha t}\right)\cdot\left(\frac{1}{n}\right)\cdot\exp\left\{-\frac{x^2}{2\alpha t} - \frac{(|x_0|+\delta_0)^2}{2\alpha t} - \varepsilon\right\}. \tag{6.11}$$

Recall that the N-dimensional normalized Brownian motion $\{\boldsymbol{\beta}(t): t \geqslant 0\}$ consists of N independent one-dimensional normalized Brownian motions, i.e.

$$\boldsymbol{\beta}(t) = (\beta_1(t),\ldots,\beta_N(t))$$

where $\{\beta_j(t): t \geqslant 0\}$ is a one-dimensional normalized Brownian motion. Let $x_0 = (x_{01},\ldots,x_{0N})$, $x = (x_1,\ldots,x_N) \in \mathbf{R}^N$. Then

$$\begin{aligned}P_x(S_n) &= P_x\{|\boldsymbol{\beta}(t) - x_0| \leqslant \delta_0, |\boldsymbol{\beta}(s)| < 1/n \text{ for } s \in J = [\alpha t, (1-\alpha)t]\}\\ &\geqslant P_x\left\{\text{For } j = 1,\ldots,N, |\beta_j(t) - x_{0j}| \leqslant \delta_0/\sqrt{N}, |\beta_j(s)| < 1/n\sqrt{N} \text{ for } s \in J\right\}\\ &= \left(P_x\left\{|\beta_1(t) - x_{01}| \leqslant \delta_0/\sqrt{N}, |\beta_1(s)| < 1/n\sqrt{N} \text{ for } s \in J\right\}\right)^N\end{aligned}$$

by independence

$$\geqslant K_0\exp\{-\pi^2 n^2(1-2\alpha)N^2 t/8(1-\varepsilon)\}$$

by our one-dimensional results where

$$K_0 = \left(\frac{8\alpha_1\delta_0}{\sqrt{N}\,\pi\alpha t}\right)^N\left(\frac{1}{n}\right)^N\exp\left\{-\frac{|x|^2}{2\alpha t} - \frac{(N|x_0|^2 + \delta_0^2)}{\alpha t} - \varepsilon\right\}. \quad \square \tag{6.12}$$

In [2] we indicate how ideas from this section can be combined with the L^2 theory of $-\Delta + c/|x|^2$ developed by quantum theorists to give a complete proof of Theorem 2.2(ii) for the case of $f \equiv 0$ and $\Omega = \mathbf{R}^N$. Incidently, [2] contains a list of some of the L^2 references.

7. Concluding remarks.

REMARK 7.1. The potential $V(x) = c/|x|^2$ is of the form $V = -\Delta\phi/\phi$ where $\phi(x) = |x|^{-\alpha}$ and $c = (N-2-\alpha)\alpha$. Theorem 2.2 extends to cover potentials of the form $V = -\Delta\phi/\phi$ where $\phi > 0$, $\Delta\phi \in L^1_{\text{loc}}(\Omega)$, $\int \phi u_0 < \infty$, $\int_0^T \int_\Omega f\phi < \infty$ and the weighted Sobolev estimate

$$\left(\int_\Omega |h|^m \phi\right)^{1/m} \leqslant C\left(\int_\Omega |\nabla h|^2\phi^2 + \int_\Omega |h|^2\phi^2\right)^{1/2}$$

holds for some $m > 2$. (This last estimate reduces to (4.12) for $\phi(x) = |x|^{-\alpha}$ and for h a radial function.)

REMARK 7.2. Theorem 2.2(ii) extends to establish the nonexistence of solutions for certain nonlinear problems. For instance, consider (P) in which the term $V(x)u$ is replaced by the nonlinear term $W(x,u)u$, where $W(x,u) \geqslant c|x|^2$ holds for all $(x,u) \in \Omega \times \mathbf{R}$ and $c > C_*(N)$. Then no nonnegative solution to (P) can exist in that the approximating solutions u_n of (P_n) approach infinity on all of $\Omega \times]0,T]$.

REMARK 7.3. It is well known that the solution of the Cauchy problem $\partial u/\partial t - \Delta u = 0$, $u(x,0) \equiv 0$ is not unique (for $(x,t) \in \mathbf{R}^N \times]0,\infty[$). Nonuniqueness also holds for (P). This will be shown by one of us in a separate paper [1].

Appendix: Proof of inequality (4.12). If $0 \leqslant h \in C^1[0,2r]$ and $h(2r) = 0$, then we claim that

$$\text{(A.1)} \qquad \left(\int_0^{2r} h^m(s)s^{a-1}\,ds\right)^{1/m} \leqslant K_0\left(\int_0^{2r}\left|\frac{dh}{ds}(ds)\right|^2 s^{a-1}\,ds\right)^{1/2}$$

where $a = N - 2\alpha > 2$, $m^{-1} = 2^{-1} - a^{-1}$, and K_0 is a constant, depending only on α. For the proof, integration by parts gives

$$\begin{aligned}\int_0^{2r} h^m(s)s^{a-1}\,ds &= -\frac{m}{a}\int_0^{2r} h^{m-1}(s)\,\frac{dh(s)}{ds}s^a\,ds \\ &\leqslant \frac{m}{a}\left(\int_0^{2r} h^{2m-2}s^{a+1}\,ds\right)^{1/2}\left(\int_0^{2r}\left(\frac{dh}{ds}\right)^2 s^{a-1}\,ds\right)^{1/2} \\ &\leqslant \frac{m}{a}\sup_{s\in[0,2r]}\{s^2h^{m-2}(s)\}\left(\int_0^{2r} h^m s^{a-1}\,ds\right)^{1/2}\left(\int_0^{2r}\left|\frac{dh}{ds}\right|^2 s^{a-1}\,ds\right)^{1/2}.\end{aligned}$$

Thus to establish (A.1) it suffices to show

$$\text{(A.2)} \qquad \sup_{[0,2r]}\{s^2h^{m-2}(s)\} \leqslant K_1\left(\int_0^{2r}\left(\frac{dh}{ds}\right)^2 s^{a-1}\,ds\right)^{m-2/2},$$

from which it follows that (A.1) holds with $K_0 = [K_1 m^2/a^2]^{1/m}$ which depends only on α.

For (A.2) we note that $2/(m-2) = a/m$, and so

$$\begin{aligned}\sup_{[0,2r]} \{s^{2/(m-2)}h(s)\} &= \sup_{[0,2r]} \{s^{a/m}[h(s) - h(2r)]\} \\ &= \sup\left\{s^{a/m}\int_s^{2r}(-h'(\sigma))\,d\sigma\right\} \\ &\leqslant \sup\left\{s^{a/m}\left(\int_s^{2r}\sigma^{1-a}\,d\sigma\right)^{1/2}\left(\int_s^{2r}h'(\sigma)^2\sigma^{a-1}\,d\sigma\right)^{1/2}\right\} \\ &\leqslant \left\{\sup_{[0,2r]} M(s)\right\}\left(\int_0^{2r}h'(\sigma)^2\sigma^{a-1}\,d\sigma\right)^{1/2}\end{aligned}$$

where $M(s) = s^{a/m}(\int_s^{2r}\sigma^{1-a}\,ds)^{1/2}$. On $(0, 2r)$ we have

$$\begin{aligned}M^2(s) &= s^{2a/m}\left((2r)^{2-a} - s^{2-a}\right)(2-a)^{-1} \\ &= \left((2r)^{2-a}s^{2a/m} - 1\right)(2-a)^{-1}\end{aligned}$$

since $2a/m + 2 - a = 0$; hence

$$\frac{d}{ds}M^2(s) = (2r)^{2-a}(2-a)^{-1}\frac{2a}{m}s^{2a/m-1} < 0$$

because $a > 2$. Thus $\sup_{[0,2r]} M^2(s) = M^2(0) = (a-2)^{-1}$, and so (A.2) holds with $K_1 = (a-2)^{-(m-2)/2}$.

Next we deduce (4.12). Fix $\rho > 0$ and let $r \geqslant \rho$. Let $h \in C^1(0, r)$. Let $\xi \in C^1[r, 2r]$ satisfy $0 \leqslant \xi \leqslant 1$, $\xi \equiv 0$ in $[r + \rho/2, 2r]$, $\xi \equiv 1$ in $[r, r + \rho/4]$, and $0 \geqslant \xi' \geqslant -5/\rho$ in $[r, 2r]$. Let $\psi(s)$ be $h(s)$ or $h(2r - s)\xi(s)$ according as $s \in [0, r)$ or $s \in [r, 2r]$. Then by (A.2),

$$\begin{aligned}\left(\int_0^r h^m(s)s^{a-1}\,ds\right)^{2/m} &\leqslant \left(\int_0^{2r}\psi^m(s)s^{a-1}\,ds\right)^{2/m} \leqslant K_1^2\left(\int_0^{2r}\psi'(s)^2 s^{a-1}\,ds\right) \\ &\leqslant K_1^2\left[\int_0^r h'(s)^2 s^{a-1}\,ds + 2\int_r^{2r}h'(2r-s)^2\xi(s)^2 s^{a-1}\,ds\right. \\ &\qquad \left. + 2\int_r^{2r}h(2r-s)^2\xi'(s)^2 s^{a-1}\,ds\right] \\ &\qquad\qquad (\text{let } \sigma = 2r - s) \\ &\leqslant K_1^2\left[\int_0^r h'(s)^2 s^{a-1}\,ds + 2\int_{r-\rho/2}^{r}h'(\sigma)^2(2r-\sigma)^{a-1}\,d\sigma\right. \\ &\qquad \left. + 2\int_{r-\rho/2}^{r-\rho/4}h(\sigma)^2\xi'(2r-\sigma)^2(2r-\sigma)^{a-1}\,d\sigma\right] \\ &\leqslant K_1^2\left[1 + 2\cdot\left(\frac{r+\rho/2}{r-\rho/2}\right)^{a-1}\right]\int_0^r h'(s)^2 s^{a-1}\,ds \\ &\qquad + 50\rho^{-2}K_1^2\left(\frac{r+\rho/2}{r-\rho/2}\right)^{a-1}\int_0^r h(s)^2 s^{a-1}\,ds \\ &\leqslant K_2\int_0^r h'(s)^2 s^{a-1}\,ds + K_3\int_0^r h(s)^2 s^{a-1}\,ds\end{aligned}$$

where

$$K_2 = \left[(a-2)^{2-m} m^2/a^2\right]^{2/m}(1+2^a)\rho^{-2} \equiv K_2(\alpha,\rho), \tag{A.3}$$

$$K_3 = 25\left[(a-2)^{2-m} m^2/a^2\right]^{2/m} 2^a \rho^{-2} \equiv K_3(\alpha,\rho). \tag{A.4}$$

This proves (4.12). Now we study K_2 and K_3 as functions of α. (Note that these are dependent on ρ.) Since $m < \infty$ we have $\alpha < (n-2)/2$. For $\alpha \to (N-2)/2$ we observe the following. Write $\alpha = (N-2)/2 - \varepsilon$, $\varepsilon > 0$. Then $a = N - 2\alpha = 2 + 2\varepsilon$, $m = 2a/(a-2) \approx 2/\varepsilon$, and

$$\left[(a-2)^{2-m} m^2/a^2\right]^{2/m} \approx \left[(2\varepsilon)^{2-2/\varepsilon}\varepsilon^{-2}\right]^{\varepsilon} \approx 2^{2\varepsilon-2}\varepsilon^{2\varepsilon-2-2\varepsilon} = \tfrac{1}{4}\varepsilon^{-2}.$$

Thus each of K_2 and K_3 behaves like

$$\text{Const}\ \rho^{-2}[((N-2)/2)-\alpha]^{-2}$$

as $\alpha \to (N-2)/2$. Thus K_2 and K_3 blow-up in an inverse square manner in each variable.

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